K-theoretic Donaldson invariants via instanton counting

(with L. Göttsche, K. Yoshioka)

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Professor Akihiro Tsuchiya taught me

mathematicians and physists are very different,
though both study the same things.

Mathematicians : Start with <u>definitions</u>.

Physists : Start with <u>computations</u>.

Today : K-theoretic Donaldson invariants
can be computed in various examples,
but no satisfactory rigorous definition ....

§0. Introduction

Invariants based on moduli spaces (e.g. Donaldson invariants, Gromov-Witten invariants)

are usually defined via homology groups

Hope

One can define invariants for generalised homology theories

(K-homology, elliptic homology, ...)

Y. P. Lee : Quantum K-theory = K-theoretic Gromov-Witten invariants

cf. Coates-Givental

X : projective mfd

$\mathcal{M}_{g,n}(X;\beta)$ = moduli of stable maps

$\chi(\mathcal{M}^{\mathrm{vir.}}_{g,n}(X;\beta), \mathcal{L})$ = holomorphic Euler characteristic of virtual structure sheaf twisted by a line bdle

Rem. ① Givental - Lee : Quantum K-theory for flag manifolds

(earlier than Y.P. Lee)

quantum cohomology for flag $\underset{\text{Kim}}{\rightsquigarrow}$ Toda lattice

quantum K-theory $\rightsquigarrow$ difference Toda

② analogy

homology ~ Yangian

K-theory ~ quantum enveloping algebra

elliptic cohomology ~ elliptic quantum group

(classification of Yang-Baxter equ.)

Motivation

- Better understanding of invariants ?

  cf. K-theory v.s. homology $\to$ Riemann-Roch, Atiyah-Singer index thm

- integrality (inv. is $\mathbb{Z}$-valued)

- motivation from computation

  - Geometric Engineering Katz, Klemm, Vafa

    "Donaldson invariants" for $\mathbb{R}^4$

    $=$ $\boxed{\text{limits}}$ of Gromov-Witten invariants of local Calabi-Yau 3-folds

    e.g. $K_{\mathbb{P}^1 \times \mathbb{P}^1}$

    The genuine GW inv. $=$ K-theoretic Donaldson invariants

- strange duality (Le Potier)

X : projective surface ($\mathbb{P}^2$) with an ample line bundle H

$u, v \in K_{top}(X)$ s.t. $\langle u, v \rangle = 0$

$M_H(u)$ = moduli space of H-semistable torsion-free sheaves E with $[E] = u$

$\mathcal{L}_v$ = determinant line b'dle associated with v

"Conj" $\chi(M_H(u), \mathcal{L}_v) \overset{?}{=} \chi(M_H(v), \mathcal{L}_u)$

(not precise enough

(cf. level-rank duality for WZW model

Goal

- Define K-theoretic Donaldson invariants for projective surfaces

- Computation of invariants via instanton counting
  - blowup formula
  - wall-crossing formula
  - (- formula via Seiberg - Witten invariants)

§1.

X : projective surface

$v \in K_{top}(X)$

$M_H(v)$ = moduli space of H-semistable coherent sheaves E on X

(Gieseker – Maruyama) $\qquad [E] = v$

$\mathcal{E}$: universal family on $X \times M_H(v)$

$$\begin{tikzcd} & X \times M_H(v) \arrow[dl, "q"'] \arrow[dr, "p"] & \\ X & & M_H(v) \end{tikzcd}$$

$$\begin{tikzcd} K(X) \arrow[r, "q^*"] \arrow[rrrr, bend right, "\lambda_{\mathcal{E}}"'] & K^0(X \times M_H(v)) \arrow[r, "\otimes[\mathcal{E}]"] & K^0(X \times M_H(v)) \arrow[r, "p_!"] & K^0(M_H(v)) \arrow[r, "\det"] & \mathrm{Pic}(M_H(v)) \end{tikzcd}$$

$\lambda_{\mathcal{E}}(u)$ well-defined and independent of the choice of $\mathcal{E}$

if $\quad \chi(X, u \otimes v) = 0$

Assume $M_H(v)$ is <u>of expected dimension</u>.

Define K-theoretic Donaldson invariant

$\chi(M_H(v), \lambda_{\mathcal{E}}(u))$ $\qquad$ holomorphic Euler characteristic

Rem. If $c_2(v) \gg 0$, $M_H(v)$ is <u>of expected dim</u>.

(Donaldson, O'Grady, ...)

In general, there is no rigorous definition so far, but we have two approaches:

① Use blowup formula

$$\begin{array}{ccc} p : \hat{X} & \to & X \\ \cup & & \cup \\ C & \to & p \end{array}$$

blowup at a point

exceptional curve

$\hat{H} := p^*H - \varepsilon C$

ε : sufficiently small

Conj. $\chi(\hat{M}_{\hat{H}}(p^*v), \lambda(p^*u)) = \chi(M_H(v), \lambda(u))$

True if $\hat{M}_{\hat{H}}(p^*v)$ is smooth e.g. $\langle -K_X, H \rangle > 0$

If we blowup at sufficiently many points, $\hat{M}_{\hat{H}}$ is of expected dimension

Use the above formula to define $\chi(M_H(v), \lambda(u))$.

② virtual structure sheaf

Suppose $M_H(v)$ consists of <u>stable</u> sheaves only. $(\Rightarrow H^0(\mathrm{End}_0 E) = 0 \ \forall E)$

$\exists$ perfect obstruction theory (Thomas)

$\longrightarrow$ virtual structure sheaf $\mathcal{O}^{\mathrm{virt}}$. can be defined

$\rightsquigarrow$ $\chi(M_H(v), \mathcal{O}^{\mathrm{virt.}} \otimes \mathcal{X}(u))$ can be considered

In general, we should put additional structures (e.g. parabolic str.) to construct a perfect obstruction theory and prove the independence from the additional structure.

(Such a construction exists for homological version T. Mochizuki)

<u>Two approaches should give the same definition.</u>

metric dependence

- $p_g > 0 \Rightarrow$ invariants are expected to be independent of H.

  (follows from T. Mochizuki's theory)

- $p_g = 0 \Rightarrow$ wall-crossing formula

  rk $v = 2$

  $\mathcal{C}$ = ample cone

  $\xi \in H^2(X, \mathbb{Z}) \setminus \{0\}$ $\qquad W^\xi := \{x \in \mathcal{C} \mid \langle x, \xi \rangle = 0\}$ wall

  ξ is of type $v \iff$ 1) $\xi + c_1(v) \equiv 0 \mod 2$

  2) d = exp. dim for v $\qquad d + 3 + \xi^2 \geqq 0$

  chamber = component of $\mathcal{C} \setminus \bigcup_{\xi : \text{type } v} W^\xi$ (finite)

  invariant depends only on a chamber

If H_1, H_2 are separated by W^ξ, $\chi(M_{H_1}(v), \lambda(u)) - \chi(M_{H_2}(v), \lambda(u))$

can be given by a holomorphic Euler characteristic of

a virtual vector bundle on the product of Hilbert schemes:

$$\Delta_{\xi,T}(u,\Lambda) := \sum_{n,m\geq 0} \frac{\Lambda^d}{T^{\left(-\langle \frac{\xi}{2}, u^{(1)}\rangle + \frac{\mathrm{rk}\, u}{2}(n-m)\right)}} \chi\left(X^{[n]}\times X^{[m]}, \frac{\lambda_{\Xi_1}(u)\otimes\lambda_{\Xi_2}(u)}{\Lambda_{-T}\mathcal{A}_{\xi,+}^\vee \, \Lambda_{-T^{-1}}\mathcal{A}_{\xi,-}^\vee}\right)$$

$$d := 4(n+m)+\xi^2-3$$

$$u^{(1)} = c_1(u) + \frac{\mathrm{rk}\, u}{2}(c_1(v)-K_X)$$

$$\begin{tikzcd} & X\times X^{[n]}\times X^{[m]} \arrow[dl, "q"'] \arrow[dr, "p"] & \\ X & & X^{[n]}\times X^{[m]} \end{tikzcd}$$

$$\Xi_1 = \mathcal{I}_{X^{[n]}}\left(\frac{c_1(v)+\xi}{2}\right)$$
$$\Xi_2 = \mathcal{I}_{X^{[m]}}\left(\frac{c_1(v)-\xi}{2}\right)$$
universal families

$$\mathcal{A}_{\xi,+} = -p_!(\mathcal{I}_{X^{[m]}}^\vee \otimes \mathcal{I}_{X^{[n]}}\otimes q^!\xi)$$
$$\mathcal{A}_{\xi,-} = -p_!(\mathcal{I}_{X^{[n]}}^\vee \otimes \mathcal{I}_{X^{[m]}}\otimes q^!\xi^\vee)$$

$$\chi(M_{H_1}(v),\lambda(u)) - \chi(M_{H_2}(v),\lambda(u))$$
$$= \text{Coeff of } \Lambda^d \text{ in } \left(\left[\text{Coeff. of } T^0 \text{ in } \Delta_{\xi,T}(u,\Lambda)\right] - \left[\text{Coeff. of } (T^{-1})^0 \text{ in } \Delta_{\xi,T}(u,\Lambda)\right]\right)$$

<u>NB.</u> Riemann-Roch $\Rightarrow$ RHS $\in \mathbb{Q}(T)[[\Lambda]]$

§2. Instanton counting (K-theoretic version)

<u>Prop.</u> $\Delta_{\mathfrak{z},\mathfrak{p}}(u,\Lambda)$ is "universal".

It is determined by $\mathfrak{z}^2$, $\mathfrak{z}\cdot K_X$, K_X^2, $c_2(X)$, $\mathfrak{z}\cdot v^{(1)}$, $K_X v^{(1)}$, $(v^{(1)})^2$

<u>Cor.</u> $\Delta_{\mathfrak{z},\mathfrak{p}}(u,\Lambda)$ can be explicitly given if we know it for $X =$ toric surface proj.

We can use Atiyah-Bott-Lefschetz fixed point formula for a computation when $X =$ toric surface

$$(X^{[n]})^T = ?$$

Z :
- supported on X^T
- monomial ideal in the toric coordinate around a point in X^T

$\rightsquigarrow$ It is enough to determine "local contribution" for each fixed point.

Formulation on $X = \mathbb{C}^2$

$M(n) :=$ framed moduli space of $\overbrace{\text{torsion-free}}^{\text{rank } 2}$ sheaves on $\mathbb{P}^2 = \mathbb{C}^2 \cup \ell_\infty$

$$= \{ (E, \Phi) \mid E|_{\ell_\infty} \xrightarrow[\cong]{\Phi} \mathcal{O}_{\ell_\infty}^{\oplus 2}, \ c_2(E) = n \} / \text{isom.}$$

$$\curvearrowleft \underbrace{\mathbb{C}^* \times \mathbb{C}^*}_{\text{acts on } \mathbb{C}^2} \times \underbrace{\mathbb{C}^*}_{\text{change of } \Phi}$$

$$\mathcal{O}_{\ell_\infty}^{\oplus 2} \longrightarrow \mathcal{O}_{\ell_\infty}^{\oplus 2} \qquad \begin{bmatrix} T & 0 \\ 0 & T^{-1} \end{bmatrix}$$

$$Z_m^{\text{inst.}} := \sum_{n=0}^{\infty} \Lambda^{4n} e^{-\beta(2+m)\frac{\varepsilon_1+\varepsilon_2}{2} \cdot n} \underbrace{\chi(M(n), \mathcal{L}^{\otimes m})}_{\uparrow} \qquad \mathcal{L} = \lambda(\mathcal{O}_{\mathbb{P}^2}(-\ell_\infty))$$

$e^{\beta\varepsilon_1}, e^{\beta\varepsilon_2}$ character of $\mathbb{C}^* \times \mathbb{C}^* \times \underbrace{\mathbb{C}^*}_{T}$

(Nekrasov 5D partition function with Chern-Simons term.)

$\Delta_{3,p}(u, \Lambda)$ (more precisely its equivariant version)

can be expressed in terms of Z_m where $m = -\mathrm{rk}\, u$

By the localization formula (Atiyah-Bott-Lefschetz type), Z_m^{inst} can be expressed in terms of Young diagrams! (purely combinatorial expression)

Nekrasov's conjecture & its refinement : $(|m| \leq 2$ ~~...~~ $)$

$$Z_m^{inst} = \exp\left[\frac{1}{\varepsilon_1\varepsilon_2}\left(F^{inst} + (\varepsilon_1+\varepsilon_2)H + \varepsilon_1\varepsilon_2 A + \frac{\varepsilon_1^2+\varepsilon_2^2}{2}B + \cdots \right)\right]$$

Seiberg-Witten prepotential

SW curve

explicitly given in terms of modular forms ass. with SW curve

$$\tilde{\Delta}_{3,P}(u,\Lambda) \text{``}=\text{''} \prod_{p_i \in X^T} Z_{-rk\,u}\left(\begin{matrix}\varepsilon_1 = w(x_i) \\ \varepsilon_2 = w(y_i)\end{matrix}, T_y e^{-\frac{i_{p_i}^* \xi}{2}} ; \Lambda e^{-\frac{\beta}{4} i_{p_i}^* (K_X + c_1(u) + \frac{rk\,u}{2}(c_1(r) - K_X))}\right)$$

$w(x_i), w(y_i)$: weights of Torus action on $T_{p_i}X$

with correction term (perturbation)

$$\Delta_{3,P}(u,\Lambda) = \lim_{\varepsilon_1,\varepsilon_2 \to 0} \tilde{\Delta}_{3,P}(u,\Lambda)$$

this can be written via modular forms when $rk\,u = 0$

Nekrasov's conjecture & its refinement.

the same conjecturally works if $|rk\,u| \leqq 2$

<u>Rem.</u> $|rk\,u| > 2$ No Seiberg-Witten curve different from $|rk\,u| \leqq 2$ case